ISRAEL JOURNAL OF MATHEMATICS 119 (2000), 291–324

PRECISE DECAY RATE ESTIMATES FOR TIME-DEPENDENT DISSIPATIVE SYSTEMS

BY

PATRICK MARTINEZ

Département de Mathématiques, E. N. S. Cachan
Antenne de Bretagne, Campus de Ker Lann, 35 170 Bruz, France
e-mail: martinez@bretagne.ens-cachan.fr

ABSTRACT

We consider the wave equation damped with a nonlinear time-dependent distributed dissipation. By generalizing a method recently introduced to study autonomous systems, we show that the energy of the system decays to zero with an explicit and precise decay rate estimate under sharp assumptions on the feedback. Then we prove that our estimates are optimal for the problem of the one dimensional wave equation damped by a nonlinear time-dependent boundary feedback. This extends and improves several earlier results of E. Zuazua and M. Nakao, and completes strong stability results of P. Pucci and J. Serrin.

1. Introduction and main results

Let Ω be a bounded open domain of class C^2 in $\mathbb{R}^N$. Let ρ: $\mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function differentiable on $\mathbb{R}_+ \times (-\infty, 0)$ and on $\mathbb{R}_+ \times (0, \infty)$, such that $v \mapsto \rho(t, v)$ is nondecreasing and $\rho(t, 0) = 0$. We are concerned with the decay property of the solutions of the problem

$$(1.1) \qquad \begin{cases} u'' - \Delta u + \rho(t, u') = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \\ u(0) = u^0, \quad u'(0) = u^1 \end{cases}$$

in the class

$$(1.2) \qquad u \in W^{2,\infty}_{\text{loc}}(\mathbb{R}_+, L^2(\Omega)) \cap W^{1,\infty}_{\text{loc}}(\mathbb{R}_+, H^1_0(\Omega)) \cap L^\infty_{\text{loc}}(\mathbb{R}_+, H^2 \cap H^1_0(\Omega)).$$

Received May 24, 1999

As usual, we define the energy of the solution of (1.1) by

$$E(t) = \frac{1}{2}\int_{\Omega}\Big(u'^2 + |\nabla u|^2\Big)dx.$$

1.1 Explicit decay rate estimates when the feedback has a linear growth at infinity. We assume that there exist a nonincreasing function $\sigma\colon \mathbb{R}_+ \to \mathbb{R}_+$ of class C^1 on $\mathbb{R}_+$ and a strictly increasing and odd function g of class C^1 on $[-1,1]$ such that $g(v) = v$ for all $|v| \geq 1$ and

$$\forall t \geq 0,\ \forall v \in \mathbb{R}, \quad \sigma(t)g(|v|) \leq |\rho(t,v)| \leq g^{-1}\Big(\frac{|v|}{\sigma(t)}\Big), \tag{1.3}$$

where g^{-1} denotes the inverse function of g. In particular, this implies that $v \mapsto \rho(t,v)$ has a linear growth at infinity, and that $\sigma(0) \leq 1$. We will consider more general situations on the behavior of ρ at infinity in subsection 1.4 in the special case of dimension 2.

Define

$$H(y) = \frac{g(y)}{y}. \tag{1.4}$$

Note that $H(0) = g'(0)$.

We will study the following cases:

Hyp. 1: We assume that (1.3) is satisfied, and that $g(v) = v$ for all $v \in \mathbb{R}$.

Hyp. 2: We assume that (1.3) is satisfied, and that there exists some $p > 1$ such that $g(v) = v^p$ on $[0,1]$.

Hyp. 3: We assume that (1.3) is satisfied, and that $g'(0) = 0$ and the function H is nondecreasing on $[0,\eta]$ for some $\eta > 0$. (Note that $H(0) = 0$.)

Note that Hyp. 1 is a special case of Hyp. 2 (with $p = 1$), and Hyp. 2 is a special case of Hyp. 3 (with $g(v) = v^p$ and $p > 1$).

We have the following

Theorem 1: *Assume that the function σ satisfies*

$$\int_0^{\infty} \sigma(t)\,dt = +\infty. \tag{1.5}$$

1. Under Hyp. 1, there exists a positive constant ω such that the energy of the solution u of (1.1) decays as:

$$\forall t \geq 0, \quad E(t) \leq E(0)\exp\Big(1 - \omega\int_0^t \sigma(\tau)\,d\tau\Big). \tag{1.6}$$

2. Under Hyp. 2, there exists a positive constant $C(E(0))$ depending on $E(0)$ in a continuous way such that the energy of the solution u of (1.1) decays as:

$$\forall t \geq 0, \quad E(t) \leq \left(\frac{C(E(0))}{\int_0^t \sigma(\tau) d\tau} \right)^{2/(p-1)}. \tag{1.7}$$

3. Under Hyp. 3, there exists a positive constant $C(E(0))$ depending on $E(0)$ in a continuous way such that the energy of the solution u of (1.1) decays as:

$$\forall t \geq 1, \quad E(t) \leq C(E(0)) \left[g^{-1} \left(\frac{1}{1 + \int_1^t \sigma(\tau)\, d\tau} \right) \right]^2. \tag{1.8}$$

Remarks: 1. Theorem 1 improves in several directions earlier results of M. Nakao [25], weakening the assumptions on the feedback, and obtaining in some cases better estimates (see Example 1); it completes also the strong stability results of P. Pucci and J. Serrin [29].

2. Under our assumptions on the feedback, if $(u^0, u^1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$, then (1.1) has a unique strong solution that satisfies (1.2).

3. Note that the constant ω in (1.6) does not depend on the solution u. This fact seems to be related to the linear growth of g at infinity (see subsection 1.4). Note also that the constants C that appear in (1.7) and (1.8) only depend on the initial energy, and not on the norm of the initial data in $H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$. Thus Theorem 1 allows us to give decay rate estimates for weak solutions.

4. The estimates provided by Theorem 1 are exactly the same for strong solutions of the problem of the wave equation damped by a nonlinear time-dependent boundary feedback (it is sufficient to combine the arguments used in this case with the ones used for example in [20]). The real problem in this case is in fact the problem of *existence* of weak and strong solutions. Quite recently, several authors considered this problem (and next the stabilization part) in the case where the feedback is linear, that means when $\rho(t, u') = \sigma(t)u'$, see, e.g., S. C. Quiroga de Caldas [31]. But similar results do not seem to be known when the feedback is nonlinear.

5. The hypothesis "H is nondecreasing on some $[0, \eta]$" is always satisfied on usual examples, and if it is not satisfied, it is easy to obtain a slightly less good estimate using the fact that g is increasing (see [19]).

6. Theorem 1 holds true if we consider the problem of the wave equation damped by a *locally distributed* feedback of the type

$$\rho(x, t, v) := a(x)\tilde{\rho}(t, v),$$

where the function $\tilde{\rho}$ satisfies Hyp. 1, Hyp. 2 or Hyp. 3, and the function $a\colon \overline{\Omega} \to \mathbb{R}_+$ is continuous and satisfies suitable geometrical conditions, for example $a(x) \geq \alpha > 0$ on a neighborhood of the boundary region

$$\Gamma(x_0) := \{x \in \partial\Omega, (x - x_0) \cdot \nu(x) \geq 0\}$$

(see E. Zuazua [37], M. Nakao [25]), or even weaker geometrical conditions (see [19]). The proof in this case is essentially the same (only the technical part requires a more refined application of the multiplier method).

1.2 Some typical examples. As an illustration, we apply our results to several typical cases.

Example 1: Consider

$$\rho(t, v) = \frac{1}{t^\theta} g(v)$$

with $\theta \in [0, 1]$; then Theorem 1 provides the following estimates:

a) Under Hyp. 1:

$$E(t) \leq E(0) e^{1 - \omega t^{1-\theta}} \quad \text{if } \theta \in [0, 1),$$
$$E(t) \leq E(0) \frac{e}{(\ln t)^\omega} \quad \text{if } \theta = 1;$$

note that if $\theta \in [0, 1)$, M. Nakao [Nak4] obtained that the energy decays faster than t^{-m} for all $m \in \mathbb{N}$.

b) Under Hyp. 2:

$$E(t) \leq \frac{C(E(0))}{t^{2(1-\theta)/(p-1)}} \quad \text{if } \theta \in [0, 1),$$
$$E(t) \leq \frac{C(E(0))}{(\ln t)^{2/(p-1)}} \quad \text{if } \theta = 1.$$

Note that our assumption (1.3) is also weaker than M. Nakao's one, even in the cases a) and b), since (1.3) allows us to have $\rho(t, \sigma(t)) = 1$, and the assumptions of M. Nakao need that $\rho(t, \sigma(t)) \to 0$ when $t \to +\infty$ should be satisfied.

c) Under Hyp. 3:

$$\text{if } \forall v \in (0, \frac{1}{2}), \quad g(v) = e^{-1/v^p}, \quad \text{for some } p > 0,$$

then

$$E(t) \leq \frac{C(E(0))}{(\ln t)^{2/p}} \quad \text{if } \theta \in [0, 1),$$
$$E(t) \leq E(0) \frac{C(E(0))}{(\ln(\ln t))^{2/p}} \quad \text{if } \theta = 1.$$

Example 2: we can also consider the case where $\rho(t,v) = \sigma(t)g(v)$ with

$$\sigma(t) = \frac{1}{t(\ln t)(\ln_2 t)\cdots \ln_p(t)}$$

for t large enough and with some $p \geq 1$. Then (1.5) is satisfied and there exists $c > 0$ such that

$$\int_0^t \sigma(\tau)\,d\tau = c + \ln_{p+1}(t).$$

Then, for instance, under Hyp. 1:

$$E(t) \leq E(0)e^{1-\omega \ln_{p+1}(t)} = E(0)\frac{e}{(\ln_p(t))^\omega}.$$

Example 3: Consider that

$$\forall v \in (0,1), \quad \rho(t,v) = v^{p(t)}$$

with $p\colon \mathbb{R}_+ \to \mathbb{R}_+$ an increasing function that goes to infinity at infinity (and ρ has suitable behavior at infinity). We need to verify if this case satisfies Hyp. 3; we see that for $t \geq 0$,

$$\forall v \in (0,1), \quad \rho(t,v) \geq e^{p(t)(1-\ln p(t))}e^{-1/v}.$$

Then we can apply Theorem 1 if

$$\int_0^{+\infty} e^{p(t)(1-\ln p(t))}\,dt = +\infty. \tag{1.9}$$

This is not true if $p(t)$ goes too fast to infinity at infinity (for example if $p(t) = t$ for all $t \geq 0$). A sufficient condition that ensures us that (1.9) is satisfied is when there exist two positive integers q and q' such that

$$e^{p(t)(1-\ln p(t))} \geq \frac{1}{q' t(\ln t)\cdots(\ln_q t)},$$

which is equivalent to

$$p(t)(\ln p(t)) - 1) \leq \ln q' + \ln t + \ln_2 t + \cdots + \ln_{q+1} t;$$

this is satisfied for instance if there exists $\theta \in [0,1)$ such that

$$\forall t \geq 2, \quad p(t) = (\ln t)^\theta,$$

and in this case we obtain that for t large enough

$$E(t) \leq \frac{c(\theta)}{(\ln(\ln t))^2}.$$

We do not know if this estimate is optimal and what can be said if $p(t) = \ln t$ for $t \geq 2$.

Next we study the *optimality* of the estimates we obtained. There are very few results of optimality. A. Haraux [8] studied the problem (1.1) in dimension 1 with a time-independent and polynomial feedback and obtained partial results. See also M. Aassila [1] for an optimality result concerning another system.

Here we generalize some recent results obtained in collaboration with J. Vancostenoble (see [34]):

1.3 OPTIMALITY OF THE PREVIOUS ESTIMATES FOR A ONE DIMENSIONAL PROBLEM. We study the problem of the one dimensional wave equation damped by a *boundary time-dependent nonlinear feedback*:

$$(1.10) \qquad \begin{cases} u'' - u_{xx} = 0 & \text{in } (0,1) \times \mathbb{R}_+, \\ u(0,t) = 0 & \text{on } \mathbb{R}_+, \\ u_x(1,t) + \sigma(t)\, g(u'(1,t)) = 0 & \text{on } \mathbb{R}_+, \\ u(0) = u^0, \quad u'(0) = u^1. \end{cases}$$

Define $V := \{u \in H^1(0,1) : u(0) = 0\}$. Choose $(u^0, u^1) \in V \times L^2(0,1)$. Then d'Alembert's formula allows us to *write* what is exactly the solution of (1.10) (see subsection 4.1). In particular we see that if $(u^0, u^1) \in H^2(\Omega) \cap V \times V$, then u satisfies (1.2). Note that to our knowledge, general results of existence and regularity of strong and weak solutions of the problem of the wave equation damped by a nonlinear and time-dependent boundary feedback in dimension N seem to be unknown, or unwritten until now; such results exist in the linear case, see, e.g., [31].

Combining the methods used in [20] to study the autonomous boundary case and to prove Theorem 1, we obtain that the energy of u satisfies (1.6) (respectively (1.7) or (1.8)) if Hyp. 1 (respectively Hyp. 2 or Hyp. 3) is satisfied. In the following we show that the reverse estimates are also true under some additionnal assumptions on the function g and for some initial data, and that the condition (1.5) is also necessary:

THEOREM 2: *1. If $\int_0^\infty \sigma(\tau)\, d\tau < +\infty$, then $(0,0)$ is no longer a global attractor of the trajectories in $V \times L^2(0,1)$.*

2. Assume that $\int_0^\infty \sigma(\tau)\, d\tau = +\infty$. Assume also that ρ satisfies Hyp. 3 and that the function g is strictly convex on $[0, \frac{1}{2}]$. Moreover, define $h := \frac{1}{2} g^{-1}$ on $[0, \frac{1}{2}]$ and assume that

$$(1.11) \qquad s \mapsto s(h'(s) - 1)$$

is increasing on $[0, \eta]$ *for some* $\eta > 0$.

Then, if the initial conditions (u^0, u^1) *are small enough, there exist some positive constant* c *and* c' *such that the energy of the solution of (1.10) satisfies for* t *large enough*

$$\forall t \geq 1,\ E(t) \geq c \left[g'^{-1} \left(\frac{1}{c' + \int_1^t \sigma(\tau)\, d\tau} \right) \right]^2. \tag{1.12}$$

Remark: It is easy to check that (1.11) is satisfied when

$$|g(s)| = |s|^p \quad \text{or} \quad |g(s)| = e^{-1/|s|^p} \quad \text{or} \quad |g(s)| = e^{-e^{1/|s|}}$$

on a neighborhood of zero; then (1.12) gives the optimality of the corresponding estimate (1.7) and the ones given in Example 1 and Example 2. More generally, we proved in [34] that the estimate (1.8) is optimal for a class of functions g including $\exp(-1/s^p)$ and $\exp(-\exp(1/s))$ in the automous case (see Proposition 3.3). We let the reader verify that this property holds true for the solutions of (1.10).

Next we study the influence of behavior of the function $v \mapsto \rho(t, v)$ at infinity on the decrease of the energy. We study a special case in dimension 2, where we prove that the decrease of *strong* solutions is governed by the behavior of g in zero. This seems to be restricted to strong solutions: indeed, we prove, on a special case in dimension 1, that, on the contrary, the behavior of the energy of *weak* solutions can be very dependent on the behavior of $v \mapsto \rho(t, v)$ at infinity.

1.4 Explicit decay rate estimates when the feedback is weak at infinity. Our goal is to measure the influence of the behavior of $v \mapsto \rho(t, v)$ at infinity on the decrease of the energy. We restrict the study to a special case: first we assume that $N \leq 2$, and

Hyp. 4: We assume that $\rho(t, y) = \sigma(t) g(v)$ where g is a function of class C^1 such that $g'(0) \neq 0$ and

$$\forall |y| \geq 1, \quad |g(v)| \leq c|v|^q \quad \text{with } q \geq 1.$$

Hyp. 5: We assume that there exists $A \in (0, +\infty]$ and a positive constant c such that

$$\begin{cases} \forall t \geq 0, \forall v \in (-A, A), & |g(v)| \leq c(1 + |v|), \\ \forall t \geq 0, \forall v \in \mathbb{R} \setminus (-A, A), & \rho_t(t, v)^2 \leq c v \rho(t, v) \rho_v(t, v). \end{cases} \tag{1.13}$$

In particular, this allows us to consider the case where the dissipation is *weak* at infinity, that means

$$\frac{\sigma(t) g(v)}{v} \longrightarrow 0 \quad \text{when } |v| \longrightarrow +\infty;$$

for example, we can consider the function

$$g(v) = \frac{v}{\sqrt{1+v^2}}.$$

Then generalizing a method introduced in [21], we prove the following

THEOREM 3: *Assume that Hyp. 4, Hyp. 5 and (1.5) are satisfied. Then given a solution u that satisfies (1.2), there exists a positive constant ω, depending on $\|(u^0,u^1)\|_{H^2(\Omega)\times H^1(\Omega)}$ such that*

$$\forall t \geq 0, \quad E(t) \leq E(0)\exp\Big(1 - \omega\int_0^t \sigma(\tau)d\tau\Big). \tag{1.14}$$

Remarks: 1. Theorem 3 improves earlier results of [25] which proved that, when $\sigma(t) = t^\theta$ with $\theta \in (-1,1)$, the energy decays faster than t^{-m} for all $m \in \mathbb{N}$, and when $\theta \in \{-1,1\}$, the energy decays faster than $(\ln t)^{-m}$ for all $m \in \mathbb{N}$.

2. In fact the weakness of g at infinity has no real effect on the decreasingness of the energy of *strong* solutions: we find the same estimate on the energy as if g would satisfy

$$\alpha|v| \leq |g(v)| \leq \beta|v| \quad \text{for all } v, \text{ with } \alpha > 0.$$

The only difference comes from the fact that the decay rate depends on $\|(u^0,u^1)\|_{H^2(\Omega)\times H^1(\Omega)}$.

3. In this theorem, even the fact that strong solutions go to zero at infinity was unknown in the case where $\sigma(t)$ is not of the type t^θ. Indeed, P. Pucci and J. Serrin considered always the case where g satisfies

$$\exists\alpha > 0, \forall|v| \geq 1, \quad |g(v)| \geq \alpha|v|.$$

4. One could also study the case where $g'(0) = 0$, combining the methods used to prove Theorem 1 and Theorem 3, and study also the problem in higher dimensions; however, it seems that the weakness of g at infinity has a real influence on the decay rate of strong solutions in higher dimensions. See also the results of M. Nakao [25] who studied carefully these problems.

5. We will use Hyp. 5 only to prove that u' is bounded in $H^1(\Omega)$ (see subsection 3.1). (M. Nakao [25] assumed only Hyp. 5 with $A = 0$ and could not consider examples like (1.15).)

6. One can prove similar estimates under the weaker assumption on the feedback:

$$\sigma_1(t)|g(v)| \leq |\rho(t,v)| \leq \sigma_2(t)|g(v)|,$$

under suitable assumptions on σ_1 and σ_2.

It is a natural question to ask what can be said about *weak* solutions. In the case of (1.1) under Hyp. 4, we have no answer; even strong stability properties seem to be unknown. But we have some answers in the case of the problem (1.10): their energy decays to zero, but the decrease can be very weak:

1.5 INFLUENCE OF THE WEAKNESS OF THE FEEDBACK ON WEAK SOLUTIONS. Once again we consider the problem (1.10) in the special case where the feedback $\rho(t,v)$ is defined by $\rho(t,v)=\sigma(t)\,g(v)$, where σ satisfies (1.5) and g is the following function:

$$\begin{cases} \forall |s|\le 2, & g(s)=s/2,\\ \forall |s|\ge 2, & g(s)=\operatorname{sgn}(s)1. \end{cases} \tag{1.15}$$

Motivated by recents works of P. Cannarsa, V. Komornik and P. Loreti [3], we also consider the sequence of iterated logarithms

$$\begin{cases} \forall t>1, & \ln_1(t)=\ln(t),\\ \forall t>T_{p+1}, & \ln_{p+1}(t)=\ln(\ln_p(t)), \end{cases} \tag{1.16}$$

where $(T_p)_p$ is defined by $\begin{cases} T_1=1,\\ T_{p+1}=e^{T_p}. \end{cases}$

The functions $\ln_p$ are well defined on $[T_p,+\infty[$ and go slowly to infinity at infinity.

About the automous case, motivated by several remarks of A. Haraux and F. Conrad concerning the same question, we proved that *strong* solutions decay exponentially to zero, but *weak* solutions can decay *very slowly* to zero (see [34]). Quite the same results hold in the time-dependent case. More precisely, we have the following

THEOREM 4: *Assume that (1.5) is satisfied.*

1. Let g be the function defined by (1.15) and $\rho(t,v)=\sigma(t)g(v)$. Given $(u^0,v^0)\in W^{1,\infty}(0,1)\times L^\infty(0,1)$, there exists ω that depends on $\|(u^0,v^0)\|_{W^{1,\infty}(0,1)\times L^\infty(0,1)}$ such that the energy of the strong solution of (1.10) satisfies

$$\forall t\ge 0,\quad E(t)\le E(0)\exp\Big(1-\omega\int_0^t \sigma(\tau)\,d\tau\Big). \tag{1.17}$$

2. The energy of weak solutions decreases to zero at infinity.

3. Given $p\ge 1$, there exist $(u^0,u^1)\in V\times L^2(0,1)$, such that the energy of the associated solution u of (1.10) satisfies, for t large enough:

$$E(t)\ge \frac{1}{\ln_p(\int_0^t \sigma(\tau)\,d\tau)}. \tag{1.18}$$

Remarks: 1. This result allows us to measure the gap between the decrease of the energy of strong solutions and that of weak solutions. Therefore, when the feedback is weak at infinity, the behavior of g at infinity has much effect on the decrease of the energy.

2. We conjecture that the following stronger result is also true: given $f\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ a decreasing function that goes to zero at infinity, there exist $(u^0, v^0) \in V \times L^2(0,1)$ and $\varepsilon > 0$ such that the energy of the associated solution u of (1.10) satisfies for t large enough

$$E(t) \geq \varepsilon f(t).$$

3. The proof of (1.18) is based on the construction of explicit special initial conditions.

4. The proof of strong stability is not relied on for the special function g, and the result holds true if g is an odd and increasing function of class C^1 on $[-1,1]$, without any restriction on its growth at infinity.

1.6 Relation to literature. This problem has been widely studied when σ is constant on $\mathbb{R}_+$ and when ρ satisfies Hyp. 2. E. Zuazua [35] proved that the energy of the solution of (1.1) decays exponentially if $p = 1$ and in a polynomial way if $p > 1$: in this case, there exists some positive constant C such that

$$\forall t \geq 0, \quad E(t) \leq \frac{C}{(1+t)^{2/(p-1)}}.$$

See also among others [1, 4, 5, 9, 11, 12, 22, 24, 32].

When the function ρ is time-independent and weaker than any polynomial in zero (see, e.g., Example 1 c)), we provided *explicit* decay rate estimates in a recent work [19]. Our result completed a work of I. Lasiecka and D. Tataru [15], who studied the problem of the wave equation damped by a nonlinear and time-independent boundary feedback and proved that, under weak geometrical assumptions, the energy decays faster than the solution $S(t)$ of some associated ordinary differential equation $S'(t) + q(S(t)) = 0$, where the function q is increasing and depends on ρ through some algorithm. See also the related works [10, 13, 14].

However, several mathematical physics models, such as the telegraphic equation or the damped Klein–Gordon equation, are really time-dependent, and it is interesting to study what happens if, for instance, the effect of the dissipation weakens on and on as time goes by. The problem of strong stability for the solutions of (1.1) have been deeply investigated by P. Pucci and J. Serrin [27, 28, 29, 30] under very general conditions; they proved that the energy decays to zero if the damping term $\rho(t,u')$ satisfies some integral condition that prevents it from being either too small (*underdamping*) or too large (*overdamping*) when t goes to infinity. In particular, under (1.3), the rather natural hypothesis that ensures us that the energy goes to zero is (1.5): first they proved in [29] (see Corollary 5.3 p. 202) that under this additional condition, $(0,0)$ is a global attractor of the trajectories in $H_0^1(\Omega)\times L^2(\Omega)$; next they considered the linear problem

$$\begin{cases} u''-\Delta u+t^{\alpha}(\ln t)^{\beta}u'+V(x)u=0 & \text{in } \Omega\times\mathbb{R}_+,\\ u=0 & \text{on } \partial\Omega\times\mathbb{R}_+,\\ u(0)=u^0, & u'(0)=u^1, \end{cases}$$

with some potential $V(x)$ and they provided solutions that do not go to zero, if $\alpha=-1$ and $\beta<-1$, or if $\alpha<-1$ (in these cases, there exist oscillating solutions), or in the symmetric cases: if $\alpha=1$ and $\beta>1$, or if $\alpha>1$ (in these cases, there exist solutions that converge to a non-zero state).

More recently, M. Nakao [25] studied the decay of the *strong solutions* of (1.1) in the special case where

$$\begin{cases} |\rho(t,v)| & \text{behaves like } t^{\theta}|v|^{p} \text{ on } [-1,1],\\ |\rho(t,v)| & \text{behaves like } t^{\theta}|v|^{q} \text{ on } \mathbb{R}\setminus[-1,1], \end{cases}$$

where $p>0$, $0<q<q(N)$, and obtained explicit decay estimates, depending on p, q and θ; note that in the particular case where $q=1$, i.e. when $v\mapsto\rho(t,v)$ has a linear growth at infinity, the restrictions made on θ impose that, with our notations, (1.5) is always satisfied.

The paper is organized as follows: Section 2 contains the proof of Theorem 1; it is based on the construction of a special weight function ϕ and on the generalization of a technique of partition of the boundary introduced by E. Zuazua (see subsections 2.3–2.6). Section 3 contains the proof of Theorem 3 (same ideas). Section 4 contains the proof of Theorem 2; it is based on d'Alembert's formula and on the study of the behavior of some real sequences. Section 5 contains the proof of Theorem 4 (same ideas).

2. Proof of Theorem 1

2.1 A NONLINEAR INTEGRAL INEQUALITY. The proof of Theorem 1 is based on the following nonlinear integral inequalities, that we already used in [18] and [20]:

LEMMA 1: *Let $E\colon\mathbb{R}_+\longrightarrow\mathbb{R}_+$ be a nonincreasing function and $\phi\colon\mathbb{R}_+\longrightarrow\mathbb{R}_+$ a strictly increasing function of class C^1 such that*

$$\phi(0)=0 \quad\text{and}\quad \phi(t)\longrightarrow+\infty \quad \text{as } t\longrightarrow+\infty. \tag{2.1}$$

Assume that there exist $q \geq 0$, $q' \geq 0$, $c \geq 0$ and $\omega > 0$ such that

$$(2.2) \quad \forall S \geq 0, \quad \int_S^{+\infty} E(t)^{1+q}\phi'(t)dt \leq \frac{1}{\omega}E(S)^{1+q} + \frac{c}{(1+\phi(S))^{q'}}E(0)^q E(S).$$

Then E has the following decay property: if $q = 0 = c$, then

$$(2.3) \qquad \forall t \geq 0, \ E(t) \leq E(0)e^{1-\omega\phi(t)};$$

if $q > 0$, there exists $C > 0$ such that

$$(2.4) \qquad \forall t \geq 0, \quad E(t) \leq E(0)\frac{C}{(1+\phi(t))^{(1+q')/q}}.$$

Remarks: 1. Note that if $\phi(0) \neq 0$, it is sufficient to replace $\phi(t)$ by $\phi(t) - \phi(0)$ in (2.3) and (2.4).

2. We will use (2.3) to prove (1.6) and (2.4) to prove (1.7) and (1.8).

A complete proof can be found in [18]. For the reader's convenience, we give a sketch of the proof of Lemma 1:

Sketch of the proof of Lemma 1: We introduce the nonincreasing function $f\colon [0, +\infty[\longrightarrow \mathbb{R}_+$ defined by

$$f(\tau) = E(\phi^{-1}(\tau)).$$

Then thanks to the change of variables defined by ϕ, we see that f satisfies:

$$\forall t \geq 0, \quad \int_t^{+\infty} f(\tau)^{1+q} \, dtau \leq cf(t)^{1+q} + \frac{c}{(1+t)^{q'}}f(0)^q f(t).$$

When $q' = 0$, we apply a result of V. Komornik [11]. We conclude the proof of Lemma 1 by induction on the integer part of q'.

2.2 Inequality given by the multiplier method. We note that the regularity of the solution u given by (1.2) is sufficient to justify all the computations (where we will omit to write the differential elements) that lead one to prove (1.6) (or (1.7), (1.8)).

First we need an expression for E':

Lemma 2: *The function E: $\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is nonincreasing, locally absolutely continuous and*

$$(2.5) \qquad E'(t) = -\int_\Omega u'\rho(t, u')\, dx.$$

Proof of Lemma 2: This is a well known result.

$$0=\int_S^T\int_\Omega u'(u''-\Delta u+\rho(t,u'))=\Big[\frac{1}{2}\int_\Omega u'^2+|\nabla u|^2\Big]_S^T+\int_S^T\int_\Omega u'\rho(t,u'). \qquad \blacksquare$$

The proof of Theorem 1 is based on the following inequality:

LEMMA 3: *Set $q\geq 0$. There exists a positive constant c such that for all $S<T$*

$$\int_S^T E(t)^{1+q}\phi'(t)dt\leq cE(S)^{1+q}+c\int_S^T E(t)^q\phi'(t)\int_\Omega u'^2+\rho(t,u')^2\,dx\,dt. \tag{2.6}$$

Remark: This inequality is classical when $\phi(t)=t$ for all $t\geq 0$. The function ϕ will be chosen later (ϕ will be closely related to g and σ). The proof is the same as in the autonomous case [19].

Proof of Lemma 3: We integrate by parts the expression

$$0=\int_S^T E^q\phi'\int_\Omega u\,(u''-\Delta u+\rho(t,u'))\,dx\,dt,$$

and we get that

$$\begin{aligned}2\int_S^T E^{1+q}\phi' &=\int_S^T E^q\phi'\int_\Omega u'^2+|\nabla u|^2\\ &=-\Big[E^q\phi'\int_\Omega uu'\Big]_S^T+\int_S^T(qE'E^{q-1}\phi'+E^q\phi'')\int_\Omega uu'\\ &\quad+\int_S^T E^q\phi'\int_\Omega 2u'^2-u\,\rho(t,u').\end{aligned}$$

Since E is nonincreasing and ϕ' is a bounded nonnegative function on $\mathbb{R}_+$ (and we denote by λ its maximum), we easily estimate the right-hand side terms of (2.7):

$$\begin{aligned}\Big|E(t)^q\phi'(t)\int_\Omega u'u\Big| &\leq\frac{c\lambda}{q+1}E(t)^{1+q},\\ \Big|\int_S^T qE'E^{q-1}\phi'+E^q\phi''\int_\Omega uu'\Big| &\leq c\lambda\int_S^T -E'(t)E(t)^q\,dt\\ &\quad+c\int_S^T E(t)^{1+q}(-\phi''(t))\,dt\\ &\leq c\lambda E(S)^{1+q}.\end{aligned}$$

At last we see that, given $\varepsilon > 0$, we have

$$\int_S^T E^q\phi' \int_\Omega u\,\rho(t,u') \leq \frac{1}{2}\int_S^T E^q\phi' \int_\Omega \varepsilon u^2 + \frac{1}{\varepsilon}\rho(t,u')^2$$
$$\leq c\varepsilon \int_S^T E^{1+q}\phi' + \frac{1}{\varepsilon}\int_S^T E^q\phi' \int_\Omega \rho(t,u')^2.$$

Then the estimate (2.6) follows choosing ε small enough. ∎

2.3 Proof of (1.6). We consider the case where

$$\forall t \in \mathbb{R}, \forall v \in \mathbb{R}, \quad \sigma(t)|v| \leq |\rho(t,v)| \leq \frac{1}{\sigma(t)}|v|.$$

Then we have

$$\forall t \in \mathbb{R}, \forall x \in \Omega, \quad {u'}^2 + \rho(t,u')^2 \leq \frac{2}{\sigma(t)}u'\rho(t,u'). \tag{2.8}$$

Therefore we deduce from Lemma 3 (applied with $q = 0$) that

$$\int_S^T E\,\phi'\,dt \leq CE(S) + 2C\int_S^T \phi' \int_\Omega \frac{1}{\sigma(t)}u'\,\rho(x,u')\,dx\,dt. \tag{2.9}$$

Define

$$\phi(t) = \int_0^t \sigma(\tau)\,d\tau. \tag{2.10}$$

It is clear that ϕ is a concave nondecreasing function of class C^2 on $\mathbb{R}_+$. The hypothesis (1.5) ensures that

$$\phi(t) \longrightarrow +\infty \quad \text{when } t \longrightarrow +\infty. \tag{2.11}$$

Then we deduce from (2.9) that

$$\int_S^T E(t)\,\phi'(t)\,dt \leq CE(S) + 2C\int_S^T \int_\Omega u'\rho(t,u')\,dx\,dt \leq 3CE(S). \tag{2.12}$$

Then the Gronwall type inequality (2.3) gives us that

$$E(t) \leq E(0)e^{1-\phi(t)/(3C)}. \tag{2.13}$$ ∎

2.4 Proof of (1.7). Now we assume that there exists $p > 1$ such that (1.3) is satisfied with $g(v) = v^p$ on $[0,1]$. Define the function ϕ by (2.10). We apply Lemma 3 with $q := (p-1)/2$.

We need to estimate

$$\int_S^T E^q \phi' \int_\Omega u'^2 + \rho(t,u')^2 \, dx \, dt.$$

For $t \geq 0$, consider

$$\Omega_{1,v}^t := \{x \in \Omega, |u'| \leq 1\} \quad \text{and} \quad \Omega_{2,v}^t := \{x \in \Omega, |u'| > 1\},$$
$$\Omega_{1,\rho}^t := \{x \in \Omega, |u'| \leq \sigma(t)\} \quad \text{and} \quad \Omega_{2,\rho}^t := \{x \in \Omega, |u'| > \sigma(t)\}.$$

First we note that for every $t \geq 0$,

$$\Omega_{1,v}^t \cup \Omega_{2,v}^t = \Omega = \Omega_{1,\rho}^t \cup \Omega_{2,\rho}^t.$$

Next we deduce from Hyp. 2 that for every $t \geq 0$,

$$\text{if } x \in \Omega_{1,v}^t, \text{ then } u'^2 \leq \Big(\frac{1}{\sigma(t)} u' \rho(t,u')\Big)^{2/(p+1)},$$
$$\text{if } x \in \Omega_{2,v}^t, \text{ then } u'^2 \leq \frac{1}{\sigma(t)} u' \rho(t,u'),$$
$$\text{if } x \in \Omega_{1,\rho}^t, \text{ then } \rho(t,u')^2 \leq \Big(\frac{1}{\sigma(t)} u' \rho(t,u')\Big)^{2/(p+1)},$$
$$\text{if } x \in \Omega_{2,\rho}^t, \text{ then } \rho(t,u')^2 \leq \frac{1}{\sigma(t)} u' \rho(t,u').$$

Hence, using Jensen's inequality, we get that

$$\begin{aligned} (2.14) \qquad & \int_S^T E^q \phi' \int_\Omega u'^2 + \rho(t,u')^2 \, dx \, dt \\ \leq & 2 \int_S^T E^q \phi' \int_\Omega \frac{1}{\sigma} u' \rho(t,u') \, dx \, dt + 2 \int_S^T E^q \phi' \int_\Omega \Big(\frac{1}{\sigma} u' \rho(t,u')\Big)^{2/(p+1)} dx \, dt \\ \leq & 2 \int_S^T E^q \int_\Omega u' \rho(t,u') \, dx \, dt + 2 \int_S^T E^q \phi' \Big(\int_\Omega \frac{1}{\sigma} u' \rho(t,u') \, dx\Big)^{2/(p+1)} dt \\ \leq & cE(S)^{1+q} + 2 \int_S^T E^q \phi'^{(p-1)(p+1)} \Big(\frac{-E'\phi'}{\sigma}\Big)^{2/(p+1)} dt. \end{aligned}$$

Set $\varepsilon > 0$; thanks to Young's inequality and to our definitions of q and ϕ, we obtain

$$\begin{aligned} (2.15) \qquad & \int_S^T E^q \phi' \int_\Omega u'^2 + \rho(t,u')^2 \, dx \, dt \\ \leq & cE(S)^{1+q} + 2 \frac{p-1}{p+1} \varepsilon^{(p+1)/(p-1)} \int_S^T E^{1+q} \phi' \, dt + \frac{4}{p+1} \frac{1}{\varepsilon^{(p+1)/2}} E(S). \end{aligned}$$

Therefore, choosing $\varepsilon > 0$ small enough, we deduce from Lemma 3 and (2.15) that

$$\int_S^T E^{1+q}\,\phi'\,dt \le 2CE(S),$$

and the Gronwall type inequality given by Lemma 1 (applied with $c = 0$) ensures us that

$$E(t) \le \frac{C}{\phi(t)^{2/(p-1)}}. \qquad \blacksquare$$

Now we assume that Hyp. 3 is satisfied with some strictly increasing odd function g of class C^1. We generalize the method we used to study the time-independent problem (see [19]). The key point is to construct a suitable weight function ϕ and convenient partitions of Ω. In the following, we assume that the function H is nondecreasing on $[0,1]$ (if H is nondecreasing only on $[0,\eta]$ for some $\eta > 0$, it is easy to adapt the proof, see [20] subsection 4.3).

2.5 THE DEFINITION AND THE PROPERTIES OF THE WEIGHT FUNCTION USEFUL TO PROVE (1.8). When the feedback law depends on time, we need to generalize the construction used in [19]. Assume that the function ρ satisfies Hyp. 3. Define

$$\forall t \ge 1, \quad \tilde{\psi}(t) := 1 + \int_1^t \frac{1}{H(1/\tau)}\,d\tau. \tag{2.16}$$

Then $\tilde{\psi}$: $[1,+\infty[\longrightarrow [1,+\infty[$ is a strictly increasing and convex function of class C^2 and it is clear that

$$\tilde{\psi}(t) \longrightarrow +\infty \quad \text{and} \quad \tilde{\psi}'(t) \longrightarrow +\infty \quad \text{when } t \longrightarrow +\infty.$$

Now define as in the study of the autonomous case

$$\forall t \ge 1,\ \tilde{\phi}(t) := \tilde{\psi}^{-1}(t) \tag{2.17}$$

and

$$\forall t \ge 1, \quad \phi(t) := \tilde{\phi}\Big(1 + \int_1^t \sigma(\tau)\,d\tau\Big). \tag{2.18}$$

Note that this definition generalizes the weight function we used in [19]. The function ϕ has the following properties:

LEMMA 4: *The function ϕ: $[1,+\infty[\longrightarrow [1,+\infty[$ defined by (2.18) is concave, of class C^2 and satisfies*

$$\phi(t) \longrightarrow +\infty \quad \text{when } t \longrightarrow +\infty, \tag{2.19}$$

$$\phi'(t) \longrightarrow 0 \quad \text{when } t \longrightarrow +\infty, \tag{2.20}$$

$$\forall t \ge 1, \quad \phi'(t) = \sigma(t) H\Big(\frac{1}{\phi(t)}\Big). \tag{2.21}$$

Moreover, we have for t large enough

$$\frac{1}{\phi(t)} \leq g^{-1}\Big(\frac{1}{1+\int_1^t \sigma(\tau)\,d\tau}\Big). \tag{2.22}$$

Proof of Lemma 4: All these properties are easy to verify. Denote

$$S(t) = 1 + \int_1^t \sigma(\tau)\,d\tau.$$

Since σ is a positive function of class $\mathcal{C}^1$, ϕ is a strictly increasing function of class $\mathcal{C}^2$ and

$$\phi'(t) = \sigma(t)\,\tilde{\phi}'(S(t)).$$

The decrease of σ implies that ϕ is concave. Moreover, we note that

$$\phi(t) \longrightarrow +\infty \quad \text{when } t \longrightarrow +\infty \quad \text{because } \int_1^{+\infty} \sigma(\tau)\,d\tau = +\infty,$$
$$\phi'(t) \longrightarrow 0 \quad \text{when } t \longrightarrow +\infty \quad \text{because } \tilde{\phi}'(t) \longrightarrow 0.$$

Next we remark that ϕ satisfies (2.21): indeed

$$\phi'(t) = \sigma(t)\,\tilde{\phi}'(S(t)) = \sigma(t)\,\frac{1}{\tilde{\psi}'(\tilde{\phi}(S(t)))} = \sigma(t)\,\frac{1}{\tilde{\psi}'(\phi(t))} = \sigma(t)\,H\Big(\frac{1}{\phi(t)}\Big).$$

At last we verify that ϕ satisfies (2.22): for t large enough we have

$$\tilde{\psi}(t) \leq 1 + \frac{t-1}{H(1/t)} \leq \frac{t}{H(1/t)} = \frac{1}{g(1/t)},$$

provided that $H(1/t) \leq 1$. Therefore

$$t \leq \tilde{\phi}\Big(\frac{1}{g(1/t)}\Big).$$

Thus we get that for t large enough

$$\frac{1}{\phi(t)} = \frac{1}{\tilde{\phi}\Big(1+\int_1^t \sigma(\tau)\,d\tau\Big)} \leq g^{-1}\Big(\frac{1}{1+\int_1^t \sigma(\tau)\,d\tau}\Big). \qquad \blacksquare$$

Note that as a consequence of (2.21), we see that there exists $k > 0$ such that

$$\forall t \geq 0,\ \phi'(t) \leq k\sigma(t). \tag{2.23}$$

2.6 Proof of (1.8). We estimate the terms of the right-hand side of (2.6) in order to apply the results of Lemma 1: we choose $q = 1$ and study first

$$\int_S^T E\phi' \int_\Omega u'^2 \, dx \, dt.$$

We have the following estimate:

Lemma 5: *There exists $C > 0$ such that*

$$\forall 1 \le S < T, \quad \int_S^T E\phi' \int_\Omega u'^2 \, dx \, dt \le C\, E(S)^2 + C\, \frac{E(S)}{\phi(S)}. \tag{2.24}$$

Proof of Lemma 5: Introduce

$$\forall t \ge 1, \quad h(t) = \frac{1}{\phi(t)}. \tag{2.25}$$

h is a decreasing positive function and satisfies

$$h(1) = 1 \quad \text{and} \quad h(t) \longrightarrow 0 \quad \text{as } t \longrightarrow +\infty.$$

Define for every $t \ge 1$

$$\Omega_{3,v}^t := \{x \in \Omega : |u'| \le h(t)\}, \tag{2.26}$$

$$\Omega_{4,v}^t := \{x \in \Omega : h(t) < |u'| \le h(1)\}, \tag{2.27}$$

$$\Omega_{5,v}^t := \{x \in \Omega : |u'| > h(1)\}. \tag{2.28}$$

Fix $S \ge 1$; first we look at the part on $\Omega_{5,v}^t$. We deduce from (1.3) that

$$\forall t \ge 1, \forall |v| \ge 1, \quad v^2 \le \frac{1}{\sigma(t)} v\, \rho(t, v).$$

Thus we have

$$\begin{aligned} \int_S^T E\phi' \int_{\Omega_{5,v}^t} u'^2 \, dx \, dt &\le \int_S^T E \frac{\phi'}{\sigma} \int_{\Omega_{5,v}^t} u'\, \rho(t, u') \, dx \, dt \\ &\le k \int_S^T E(-E') \, dt \le k\, E(S)^2. \end{aligned} \tag{2.29}$$

Next we look at the part on $\Omega_{4,v}^t$. Set $t \ge 1$ and $x \in \Omega_{4,v}^t$: then $|u'(x,t)| \le 1$. Thanks to the definition of h, to (2.21) and to Hyp. 2, we have

$$\phi'(t) u'^2 = \sigma(t) H(h(t)) u'^2 \le \sigma(t) H(u') u'^2 \le u'\, \rho(t, u').$$

Therefore

$$\int_S^T E\phi' \int_{\Omega_{4,v}^t} u'^2 \,dx\,dt \le \int_S^T E \int_{\Omega_{4,v}^t} u' \rho(t,u')\,dx\,dt \le E(S)^2. \tag{2.30}$$

At last we look at the part on $\Omega_{3,v}^t$:

$$\begin{aligned}\int_S^T E\phi' \int_{\Omega_{3,v}^t} u'^2 \,dx\,dt &\le \int_S^T E\phi'\Big(\int_{\Omega_{3,v}^t} h^2\,dx\Big)\,dt \\ \le |\Omega|E(S)\int_S^T \phi'(t)h(t)^2\,dt &= |\Omega|E(S)\int_S^T \phi'(t)\frac{1}{\phi(t)^2}\,dt \le |\Omega|\frac{E(S)}{\phi(S)}.\end{aligned} \tag{2.31}$$

We add (2.29)–(2.31) to conclude. ∎

Next we prove in the same way the following

LEMMA 6: *There exists $C > 0$ such that*

$$\forall 1 \le S < T, \quad \int_S^T E\phi' \int_\Omega \rho(t,u')^2\,dx\,dt \le C\,E(S)^2 + C\,\frac{E(S)}{\phi(S)}. \tag{2.32}$$

Check of the proof of Lemma 6: We use the same strategy: define for every $t \ge T_0$

$$\Omega_{3,\rho}^t := \{x \in \Omega : g^{-1}(|u'|/\sigma(t)) \le h(t)\}, \tag{2.33}$$

$$\Omega_{4,\rho}^t := \{x \in \Omega : h(t) < g^{-1}(|u'|/\sigma(t)) \le 1\}, \tag{2.34}$$

$$\Omega_{5,\rho}^t := \{x \in \Omega : g^{-1}(|u'|/\sigma(t)) > 1\}. \tag{2.35}$$

Then it is easy to verify that

$$\text{if } x \in \Omega_{5,\rho}^t, \quad \text{then } \rho(t,u')^2 \le g^{-1}(|u'|/\sigma(t))|\rho(t,u')| = |u'|/\sigma(t)|\rho(t,u')|;$$
$$\text{if } x \in \Omega_{3,\rho}^t, \quad \text{then } \rho(t,u')^2 \le h(t)^2.$$

At last we see that if $x \in \Omega_{4,\rho}^t$, then

$$\frac{\phi'(t)}{\sigma(t)} = H(h(t)) \le H\Big(g^{-1}\Big(\frac{|u'|}{\sigma(t)}\Big)\Big) = \frac{|u'|/\sigma(t)}{g^{-1}\Big(|u'|/\sigma(t)\Big)},$$

thus

$$\phi'(t)|\rho(t,u')| \le \phi'(t)g^{-1}\Big(\frac{|u'|}{\sigma(t)}\Big) \le |u'|.$$

The proof of Lemma 6 follows from these three estimates. ∎

Using Lemma 3, Lemma 5 and Lemma 6, we get that

$$\forall S \geq 1, \quad \int_S^T E(t)^2\,\phi'(t)\,dt \leq CE(S)^2 + C\frac{E(S)}{\phi(S)}. \tag{2.36}$$

Then we use Lemma 1 and the estimate (2.22) to conclude that there exist C' and $T_1 \geq 1$ such that

$$\forall t \geq T_1, \quad E(t) \leq \frac{C'}{\phi(t)^2} \leq C'\left[g^{-1}\left(\frac{1}{1+\int_1^t \sigma(\tau)}\right)\right]^2.$$

Thus the proof of Theorem 1 is achieved. ∎

3. Proof of Theorem 3

We generalize the method introduced in [21] to study the autonomous case. We consider only the case of the feedbacks of the form $\rho(t,v) = \sigma(t)g(v)$, and where Ω is a bounded domain of $\mathbb{R}^2$ (the proof is similar and simpler when $\Omega \subset \mathbb{R}$).

3.1 Bound on u' in $H^1(\Omega)\times L^2(\Omega)$. Under Hyp. 5, we prove that u' is bounded in $H^1(\Omega)$. By deriving (1.1) with respect to time, we see that $v := u'$ is a solution of the following problem:

$$\begin{cases} v'' - \Delta v + \sigma'(t)g(v) + \sigma(t)g'(v)v' = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ v = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \\ v(0) = u^1, \quad v'(0) = \Delta u^0 + \sigma(0)\rho(u^1). \end{cases}$$

Then we compute

$$\begin{aligned} 0 &= \int_0^T \int_\Omega v'(v'' - \Delta v + \sigma'(t)g(v) + \sigma(t)g'(v)v')\,dx\,dt \\ &= \left[\frac{1}{2}\int_\Omega v'^2 + |\nabla v|^2\,dx\right]_0^T + \int_0^T \int_\Omega \sigma'(t)v'g(v) + \sigma(t)g'(v)v'^2\,dx\,dt, \end{aligned}$$

thus

$$\begin{aligned} E(v)(T) - E(v)(0) = &-\int_0^T \int_\Omega \sigma'(t)v'g(v) + \sigma(t)g'(v)v'^2 \\ &\leq c\int_0^T \int_{|v|<A} 1 + v'^2 + v^2 \\ &\quad + \int_0^T \int_{|v|\geq A} \frac{-\sigma'(t)g(v)}{\sqrt{\sigma(t)g'(v)}}\sqrt{\sigma(t)g'(v)}v' - \int_0^T \int_\Omega \sigma(t)g'(v)v'^2 \\ &\leq c(|\Omega| + 2E(u)(0))(\sigma(0) - \sigma(T)) \end{aligned}$$

$$
\begin{aligned}
&+2c\int_0^T -\sigma' E(v) + c' \int_0^T \int_\Omega v\rho(t,v) \\
\leq &c_0 + 2c\int_0^T (-\sigma'(t))E(v)(t)\,dt,
\end{aligned}
$$

where c_0 is a constant that depends on $\sigma(0)$ and $E(u)(0)$. We apply Gronwall's lemma to deduce that

$$E(v)(T) \leq (E(v)(0) + c_0)e^{2c\int_0^T -\sigma'(t)\,dt} \leq (E(v)(0)+c_0)e^{2c\sigma(0)}. \quad ∎$$

In the following, we denote

$$C(u^0,u^1) := \sup\,\{\|u'\|_{H^1(\Omega)}, t \in \mathbb{R}_+\}. \tag{3.1}$$

3.2 Inequality given by the multiplier method. Set $R_0 > 0$ and define

$$\forall t \geq 0, \quad \phi(t) = \frac{1}{R_0}\int_0^t \sigma(\tau)\,d\tau. \tag{3.2}$$

R_0 will be chosen in the next subsection; it will depend on the norm of the initial conditions (u^0,u^1) in $H^2(\Omega)\times H^1(\Omega)$. We have the following

Lemma 7: *Assume that Hyp. 4 is satisfied. There exists a positive constant $c > 0$ that depends on Ω and on $E(0)$ such that the solution u of (1.1) satisfies*

$$\int_S^T E(t)\phi'(t)\,dt \leq cE(S) + c\int_S^T \phi'(t)\int_\Omega u'^2\,dx\,dt. \tag{3.3}$$

Proof of Lemma 7: We have already seen in (2.7) that

$$\int_S^T E\phi'\,dt \leq cE(S) + \int_S^T \phi' \int_\Omega 2u'^2 - u\,\rho(t,u')\,dx\,dt. \tag{3.4}$$

It remains to estimate the last term of (3.4): we have the following

Lemma 8: *There exists $c > 0$ such that for all $\varepsilon > 0$*

$$\int_S^T \phi' \int_\Omega u\,\rho(t,u')\,dx\,dt \leq \frac{c}{\varepsilon}E(S) + \frac{c}{\varepsilon}\int_S^T \phi' \int_\Omega u'^2\,dx\,dt + \varepsilon\int_S^T E\phi'\,dt. \tag{3.5}$$

Note that (3.3) follows from (3.4) and from (3.5) choosing ε small enough.

Proof of Lemma 8: There exists $\lambda' > 0$ such that

$$|g(y)| \leq \lambda'|y| \quad \text{if } |y| \leq 1.$$

Set $\eta > 0$;

$$\int_S^T \phi' \int_{|u'|\le 1} u\,\rho(t,u')\,dx\,dt \le \sigma(0)\int_S^T \phi' \int_{|u'|\le 1} \frac{\eta}{2}u^2 + \frac{1}{2\eta}g(u')^2$$
$$\le \frac{c\eta}{2}\int_S^T E\phi' + \int_S^T \phi' \int_\Omega \frac{\lambda'^2}{2\eta}u'^2.$$

Next we study the part on $|u'| > 1$: since Ω is a bounded domain of $\mathbb{R}^2$,

$$H^1(\Omega) \subset L^{q+1}(\Omega),$$

for all $q \ge 1$. Thus

$$\|u\|_{L^{q+1}(\Omega)} \le c\|u\|_{H^1(\Omega)} \le c\sqrt{E}.$$

Then

$$\Big|\int_S^T \phi' \int_{|u'|>1} u\,\rho(t,u')\,dx\,dt\Big|$$
$$\le c\int_S^T \phi'\Big(\int_\Omega |u|^{q+1}\Big)^{1/(q+1)}\Big(\int_{|u'|>1} |g(u')|^{(q+1)/q}\Big)^{q/(q+1)}$$
$$\le c\int_S^T \phi' E^{1/2}\Big(\int_{|u'|>1} u'\,g(u')\Big)^{q/(q+1)}$$
$$\le c\int_S^T \phi' E^{1/2}\Big(\frac{-E'}{\sigma}\Big)^{q/(q+1)}$$
$$\le c\eta^{q+1}\int_S^T E^{(q+1)/2}\phi' + \frac{c}{\eta^{(q+1)/q}}\int_S^T \frac{-E'}{R_0}$$
$$\le c\eta^{q+1}E(0)^{(q-1)/2}\int_S^T E\phi' + \frac{c}{\eta^{(q+1)/q}}E(S),$$

provided that $R_0 \ge 1$. We get (3.5) choosing η small enough. ∎

3.3 Proof of Theorem 3. Lemma 7 gives that

$$\int_S^T E\phi'\,dt \le cE(S) + c\int_S^T \phi' \int_\Omega u'^2\,dx\,dt.$$

It remains to estimate in a suitable way

$$\int_S^T \phi' \int_\Omega u'^2\,dx\,dt,$$

and then we will apply Lemma 1.

For each $t \geq 0$ define

$$\Omega_6^t := \{x \in \Omega : |u'| \leq R_0\}, \tag{3.6}$$

$$\Omega_7^t := \{x \in \Omega : R_0 < |u'|\}. \tag{3.7}$$

First we study

$$\int_S^T \phi' \int_{\Omega_7^t} u'^2 \, dx \, dt,$$

with the help of the following well-known interpolation result:

LEMMA 9 (Gagliardo–Nirenberg): *Let $1 \leq r < p \leq \infty$, $1 \leq q \leq p$ and $m \geq 0$. Then the inequality*

$$\|v\|_p \leq c' \|D^m v\|_q^\theta \|v\|_r^{1-\theta} \quad \textit{for } v \in W^{m,q} \cap L^r \tag{3.8}$$

holds with $c' > 0$ and

$$\theta = \left(\frac{1}{r} - \frac{1}{p}\right)\left(\frac{m}{N} + \frac{1}{r} - \frac{1}{q}\right)^{-1} \tag{3.9}$$

if $0 < \theta \leq 1$ ($0 < \theta < 1$ if $p = \infty$ and $mq = N$). (Here $\|\cdot\|_p$ denotes the usual norm of $L^p(\Omega)$.)

When $N = 2$, it follows that there exists a positive constant c that depends on Ω such that

$$\forall v \in H^1(\Omega), \quad \|v\|_{L^3(\Omega)} \leq c' \|v\|_{H^1(\Omega)}^{1/3} \|v\|_{L^2(\Omega)}^{2/3} \tag{3.10}$$

(we used (3.8) with $p = 3$, $m = 1$, $q = r = 2$, $N = 2$ and $\theta = \frac{1}{3}$).

Note that

$$\int_{\Omega_7^t} u'^2 \, dx \leq \frac{1}{R_0} \int_{\Omega_7^t} |u'|^3 \, dx \leq \frac{1}{R_0} \|u'\|_{L^3(\Omega)}^3.$$

Applying (3.10) to $v = u'$, we obtain that

$$\|u'\|_{L^3(\Omega)}^3 \leq c' \|u'\|_{H^1(\Omega)} \|u'\|_{L^2(\Omega)}^2 \leq c' \|u'\|_{H^1(\Omega)} E(t),$$

thus

$$\int_{\Omega_7^t} u'^2 \, dx \leq \frac{c'}{R_0} \|u'\|_{H^1(\Omega)} E(t) \leq \frac{c'}{R_0} C(u^0, u^1) E(t). \tag{3.11}$$

Hence we deduce from (3.5)

$$\int_S^T E\phi' \, dt \leq cE(S) + c \int_S^T \phi' \int_{\Omega_6^t} u'^2 \, dx \, dt + \frac{cc'}{R_0} C(u^0, u^1) \int_S^T E(t)\phi'(t) \, dt.$$

Choose $R_0 > 0$ such that

$$\frac{cc'}{R_0} C(u^0, u^1) \leq \frac{1}{2}. \tag{3.12}$$

Thus we get

$$\frac{1}{2} \int_S^T E\phi' \, dt \leq cE(S) + c \int_S^T \phi' \int_{\Omega_\delta^t} {u'}^2 \, dx \, dt.$$

The last term is easy to estimate since $g'(0) \neq 0$; we can choose $\alpha > 0$ such that $|g(y)| \geq \alpha |y|$ if $|y| \leq R_0$. Thus we have

$$\int_S^T \phi' \int_{\Omega_\delta^t} {u'}^2 \, dx \, dt \leq \frac{1}{\alpha R_0} \int_S^T \int_{\Omega_\delta^t} u' \rho(t, u') \, dx \, dt \leq \frac{1}{\alpha R_0} (E(S) - E(T)). \tag{3.13}$$

Finally we get

$$\frac{1}{2} \int_S^T E(t) \phi'(t) \, dt \leq cE(S) + \frac{c}{\alpha R_0} E(S) = \frac{1}{2\omega} E(S). \tag{3.14}$$

Letting T go to infinity, and applying part 1 of Lemma 1, we get that

$$E(t) \leq E(0) e^{1 - \omega \phi(t)}. \tag{3.15}$$

The proof of Theorem 3 is completed. ∎

4. Proof of Theorem 2

In this section we consider the one dimensional wave equation damped by the boundary nonlinear feedback $\sigma(t) g(u')$.

First we show how to apply d'Alembert's formula to write explicitly the solution of (1.10):

4.1 General study of the solutions. Set $A_0 \in L^\infty(-1, 1)$. Set $s \in (-1, 1)$ and consider the sequence $(A_n(s))_n$ defined by induction by the following formula:

$$A_{n+1}(s) + A_n(s) + \sigma(s + 2n + 1) \, g(A_{n+1}(s) - A_n(s)) = 0. \tag{4.1}$$

In order to simplify, we introduce the notation,

$$\forall s \in (-1, 1), \quad \sigma_n(s) := \sigma(s + 2n + 1).$$

Assume for a while that the sequence $(A_n)_n$ is well defined, and consider the absolutely continuous function $f \colon (-1, +\infty) \longrightarrow \mathbb{R}$ such that

$$\forall n \in \mathbb{N}, \quad \forall s \in (2n - 1, 2n + 1), \quad f'(s) = A_n(s - 2n).$$

Note that (4.1) implies that

$$f'(t+1)+f'(t-1)=-\sigma(t)\,g(f'(t+1)-f'(t-1))\quad \text{a.e. } t. \tag{4.2}$$

Next define the functions u^0, u^1 and u by

$$\begin{gathered}\forall x\in(0,1),\quad u^0(x):=\int_{-x}^{x}A_0(s)\,dx,\\ \forall x\in(0,1),\quad u^1(x):=A_0(x)-A_0(-x),\\ \forall(x,t)\in(0,1)\times(0,\infty),\quad u(x,t)=f(t+x)-f(t-x).\end{gathered}$$

We remark that u is the solution of the problem (1.10) with the initial conditions (u^0,u^1). (Relation (4.2) gives that $u_x(1,t)=-\sigma(t)\,g(u_t(1,t))$.) (Reciprocally, given $(u^0,u^1)\in V\times L^2(0,1)$, it is easy to construct $A_0\in L^2(0,1)$ such that the previous relations are satisfied.) We note also that the energy of u is given by

$$\begin{aligned}\forall t\geq 0,\quad E_u(2n)&=\frac{1}{2}\int_0^1(u_x^2(x,2n)+u_t^2(x,2n))\,dx\\ &=\int_{-1}^{1}f'(s+2n)^2\,ds=\int_{-1}^{1}A_n(s)^2\,ds.\end{aligned}\tag{4.3}$$

When ρ satisfies Hyp. 3, the proof leading to Theorem 1 can still be applied for the problem (1.10) and gives an upper estimate on the energy. We carefully study (4.1) to obtain a lower bound on the energy.

We will use the following properties:

LEMMA 10: *Assume that ρ satisfies Hyp. 3. Given $s\in(-1,1)$, the sequence $(A_n(s))_n$ is well defined almost everywhere, and the sequence $(\alpha_n(s):=(|A_n(s)|)_n$ is nonincreasing and converges to zero. Moreover, if $\|A_0\|_\infty$ is small enough, the sequence $(\alpha_n(s))$ satisfies*

$$\forall n\geq 0,\quad \alpha_{n+1}(s)\geq\alpha_n(s)-\sigma_n(s)\,g(2\alpha_n(s)). \tag{4.4}$$

Proof of Lemma 10: Set $s\in(-1,1)$ such that $A_0(s)\in\mathbb{R}$. Set $n\geq 0$ and assume that $A_n(s)$ is well defined. In order to prove that $A_{n+1}(s)$ is well defined, we introduce the strictly increasing function

$$\phi_n\colon y\in\mathbb{R}\mapsto y+\sigma_n(s)g(y-A_n(s));$$

ϕ_n is continuous on $\mathbb{R}$, $\phi_n(y)\longrightarrow+\infty$ when $y\longrightarrow+\infty$ and $\phi_n(y)\longrightarrow-\infty$ if $y\longrightarrow-\infty$. Thus there exists one and only one point y_{n+1} such that $\phi_n(y_{n+1})=-A_n(s)$. By definition, $A_{n+1}(s):=y_{n+1}$.

Multiplying (4.1) by $A_{n+1}(s) - A_n(s)$, we get that

$$A_{n+1}(s)^2 - A_n(s)^2 = -\sigma_n(s)\, g(A_{n+1}(s) - A_n(s))\, (A_{n+1}(s) - A_n(s)) \leq 0,$$

thus the sequence $(|A_n(s)|)_n$ is nonincreasing. Denote

$$\ell(s) := \lim{}_{n\to+\infty} |A_n(s)|;$$

we deduce from (4.3) and from Theorem 1 that

$$\int_{-1}^{1} \ell(s)^2\, ds = \lim_{t\to+\infty} E(t) = 0.$$

Thus $\ell \equiv 0$.

At last we verify (4.4): set $s \in (-1,1)$ and assume for example that $A_n(s) > 0$. We consider again the strictly increasing function ϕ_n. Since $g'(0) = 0$, there exists $\mu > 0$ such that the function $y \mapsto y - \sigma(0)\, g(2y)$ is strictly increasing on $[0,\mu]$, so in particular

$$\forall y \in]0,\mu], \quad \sigma(0)\, g(2y) < y.$$

In the following we assume that $\|A_0\|_\infty \leq \mu$. Then

$$\phi_n(0) = \sigma_n(s)\, g(-A_n(s)) \geq \sigma(0)\, g(-A_n(s)) > -A_n(s) = \phi_n(A_{n+1}(s)).$$

Therefore $A_{n+1}(s) < 0$. So the sign of the sequence $(A_n)_n$ is alternating and, since g is odd, we get from (4.1) that

$$\forall n \geq 0, \quad |A_{n+1}(s)| - |A_n(s)| = -\sigma_n(s)\, g(|A_{n+1}(s)| + |A_n(s)|),$$

thus

$$\begin{aligned} \forall n \geq 0, \quad \alpha_{n+1}(s) &= \alpha_n(s) - \sigma_n(s)\, g(\alpha_{n+1}(s) + \alpha_n(s)) \\ &\geq \alpha_n(s) - \sigma_n(s)\, g(2\alpha_n(s)) > 0. \qquad \blacksquare \end{aligned}$$

4.2 Optimality of the estimates if $\int_0^\infty \sigma(\tau)\, d\tau = +\infty$. Now we assume that $y \mapsto y(h'(y) - 1)$ is nondecreasing on $[0,\eta]$ for some $\eta > 0$. Set $A_0 \in L^\infty(-1,1)$ such that $\|A_0\|_\infty < \mu$. The functions ψ_n: $y \mapsto y - \sigma_n(s)\, g(2y)$ are nondecreasing on $[0,\mu]$. Set $s \in (-1,1)$ and define for $n \geq 0$

$$S_n(s) := \sum_{k=0}^{n} \sigma_k(s).$$

The decrease of σ implies that

$$\int_2^{2n+2} \sigma(\tau)d\tau \leq \int_{1+s}^{2n+3+s} \sigma(\tau)d\tau$$
$$\leq 2S_n(s) \leq 2\sigma_0(s) + \int_{1+s}^{2n+1+s} \sigma(\tau)d\tau \leq 2\sigma(0) + \int_0^{2n+2} \sigma(\tau)d\tau,$$

thus the sequence $(S_n)_n$ goes uniformly to infinity on $(-1,1)$.

We introduce $h = \frac{1}{2}g^{-1}$. Set $n_1 \in \mathbb{N}$ large enough, and let $(\lambda_n(s))_{n\geq 1}$ be the decreasing sequence (convergent to 0) defined by

$$\forall n \geq 1, \quad h'(\lambda_n(s)) = n_1 + S_{n-1}(s).$$

Note that for all $n \geq 1$,

$$h(\lambda_n(s)) = \frac{1}{2}(g')^{-1}\left(\frac{1}{2(S_{n-1}(s)+n_1)}\right) \longrightarrow 0 \quad \text{when } n \longrightarrow +\infty;$$

indeed,

$$h(\lambda_n(s)) \leq \frac{1}{2}(g')^{-1}\left(\frac{1}{2n_1 + \int_2^{2n} \sigma(\tau)d\tau}\right).$$

In particular,

$$h(\lambda_1(s)) \leq \frac{1}{2}(g')^{-1}\left(\frac{1}{2n_1}\right).$$

Since (4.4) and the definition of μ imply that α_1 cannot be equal to 0 on $(-1,1)$, there exist a positive constant γ and a measurable subset J of $(-1,1)$ such that $\alpha_1(s) \geq \gamma$ on J. Choose $n_1 \in \mathbb{N}$ large enough such that $h(\lambda_1(s)) \leq \gamma$ and $\lambda_1(s) \leq \eta$ on J. We prove by induction that

$$\forall n \geq 1, \forall s \in J, \quad \alpha_n(s) \geq h(\lambda_n(s)).$$

Set $s \in J$. In order to simplify the expressions, we omit writing s in the following computations. Assume $\alpha_n \geq h(\lambda_n)$. Then

$$\alpha_{n+1} \geq \alpha_n - \sigma_n\, g(2\alpha_n) \geq \psi_n(\alpha_n) \geq \psi_n(h(\lambda_n)) = h(\lambda_n) - \sigma_n\lambda_n.$$

On the other hand, there exists $\mu_n \in]\lambda_{n+1}, \lambda_n[$ such that

$$h(\lambda_n) - h(\lambda_{n+1}) = (\lambda_n - \lambda_{n+1})h'(\mu_n) \geq (\lambda_n - \lambda_{n+1})h'(\lambda_n).$$

It remains to check that

$$(\lambda_n - \lambda_{n+1})h'(\lambda_n) \geq \sigma_n\lambda_n$$

to complete the induction argument. Since the function $y \mapsto y(h'(y) - 1)$ is nondecreasing on $[0, \eta]$, we see that for n large enough

$$\begin{aligned}\lambda_n(h'(\lambda_n) - \sigma_n) - \lambda_{n+1}h'(\lambda_n) &= \lambda_n(h'(\lambda_n) - \sigma_n) - \lambda_{n+1}(h'(\lambda_{n+1}) - \sigma_n)\\ &= \lambda_n(h'(\lambda_n) - 1) - \lambda_{n+1}(h'(\lambda_{n+1}) - 1) + (1 - \sigma_n)(\lambda_n - \lambda_{n+1}) \geq 0.\end{aligned}$$

Hence

$$(\lambda_n - \lambda_{n+1})h'(\lambda_n) \geq \sigma_n \lambda_n,$$

and so $\alpha_{n+1} \geq h(\lambda_{n+1})$.

Then we easily get a lower bound on the energy: we have

$$\begin{aligned}E(2n) &= \int_{-1}^{1} \alpha_n(s)^2\, ds \geq \int_J \alpha_n(s)^2\, ds \geq \int_J h(\lambda_n(s))^2\, ds\\ &\geq |J| \left[\frac{1}{2}(g')^{-1}\left(\frac{1}{2\sigma(0) + 2n_1 + \int_0^{2n} \sigma(\tau)d\tau}\right)\right]^2,\end{aligned}$$

where $|J|$ denotes the Lebesgue measure of J. ∎

4.3 THE ENERGY DOES NOT DECAY TO ZERO IF $\int_0^\infty \sigma(\tau)\, d\tau < +\infty$. Let β_1 be a positive real number small enough such that $\beta_1 \leq \alpha_1(s)$ on J and

$$\beta_1 - g(2\beta_1) \int_0^{+\infty} \sigma(\tau)\, d\tau > 0.$$

We use the same kind of reasoning. For $s \in J$, consider the following sequence defined by induction:

$$\begin{cases} \beta_1(s) = \beta_1, \\ \beta_{n+1}(s) = \beta_n(s) - \sigma_n(s) g(2\beta_n(s)). \end{cases}$$

First we see that for all $s \in J$,

$$\forall n \geq 1, \quad \beta_n(s) \leq \alpha_n(s).$$

But since the sequence $(\beta_n(s))_n$ is nonincreasing, it converges to some nonnegative value $\ell(s)$. We claim that $\ell(s)$ is bounded from below by a positive constant on J: indeed, we have for $n \geq 1$

$$\beta_n(s) - \beta_{n+1}(s) = \sigma_n(s)\, g(2\beta_n(s)) \leq \sigma_n(s)\, g(2\beta_1(s)),$$

thus

$$\beta_1 - \ell(s) \leq g(2\beta_1) \sum_{k=1}^{+\infty} \sigma_k(s),$$

hence

$$\ell(s) \geq \beta_1 - g(2\beta_1)\sum_{k=1}^{+\infty}\sigma_k(s) \geq \beta_1 - g(2\beta_1)\int_0^{+\infty}\sigma(\tau)\,d\tau > 0.$$

Thus

$$E_u(2n) = \int_{-1}^{1}\alpha_n(s)^2\,ds \geq \int_J \alpha_n(s)^2\,ds$$
$$\geq \int_J \ell(s))^2\,ds \geq |J|\Big(\beta_1 - g(2\beta_1)\int_0^{+\infty}\sigma(\tau)\,d\tau\Big)^2. \qquad \blacksquare$$

5. Proof of Theorem 4

We keep the same notations as in Section 7. Thanks to the special choice of the function ρ, it is easy to solve (4.1), and we deduce that

$$\begin{cases} \text{if } |A_n(s)| \geq \frac{2+\sigma_n(s)}{2}, \text{ then } |A_{n+1}(s)| = |A_n(s)| - \sigma_n(s), \\ \text{if } |A_n(s)| \leq \frac{2+\sigma_n(s)}{2}, \text{ then } |A_{n+1}(s)| = k_n(s)|A_n(s)| \text{ with } k_n(s) := \frac{2-\sigma_n(s)}{2+\sigma_n(s)}. \end{cases}$$

Set $s \in (0,1)$ and let $p(s)$ be the smallest nonnegative integer such that

$$|A_0(s)| \leq S_{p(s)-1} + \frac{2+\sigma_{p(s)(s)}}{2}$$

$(p(s) = 0$ if $|A_0(s)| \leq \frac{1}{2}(2+\sigma_0(s)))$. We deduce easily that

$$\text{if } q \leq p(s), \text{ then } |A_q(s)| = |A_0(s)| - S_{q-1}(s), \tag{5.1}$$

$$\text{if } q \geq p(s)+1, \text{ then } |A_q(s)| = |A_{p(s)}(s)| \prod_{j=p(s)}^{q-1} k_j(s). \tag{5.2}$$

5.1 Decay of strong solutions. Set $(u^0, u^1) \in W^{1,\infty}(0,1) \times L^\infty(0,1)$, then $A_0 \in L^\infty(-1,1)$. Hence there exists p_0 large enough such that

$$\|A_0\|_\infty \leq \int_2^{2p_0}\sigma(\tau)\,d\tau,$$

so for all $s \in (-1,1)$ we have

$$|A_{p_0}(s)| \leq \|A_0\|_\infty \leq S_{p_0-1}(s) + \frac{2+\sigma_{p_0(s)}}{2}.$$

Hence

$$\forall n \geq p_0+1, \forall s \in (-1,1), \quad |A_n(s)| = |A_{p_0}(s)| \prod_{j=p_0}^{n-1} k_j(s) \leq \|A_0\|_\infty \prod_{j=p_0}^{n-1} k_j(s).$$

Using the fact that $\sigma(0) \leq 1$ and that

$$\forall v \in [0,1], \quad \ln\left(\frac{2-v}{2+v}\right) \leq -v,$$

we deduce that

$$\begin{aligned}
\forall n \geq p_0+1,\ E_u(2n) = \int_{-1}^{1} A_n(s)^2\,ds &\leq \|A_0\|_\infty^2 \int_{-1}^{1} \prod_{j=p_0}^{n-1} k_j(s)^2\,ds \\
&\leq \|A_0\|_\infty^2 \int_{-1}^{1} \exp\Big(2\sum_{j=p_0}^{n-1} \ln\Big(\frac{2-\sigma_j(s)}{2+\sigma_j(s)}\Big)\Big)\,ds \\
&\leq \|A_0\|_\infty^2 \int_{-1}^{1} \exp\Big(2\sum_{j=p_0}^{n-1} -\sigma_j(s)\Big)\,ds \\
&\leq \|A_0\|_\infty^2 \int_{-1}^{1} \exp\Big(-2\int_{2p_0+2}^{2n+1+s} \sigma(\tau)\,d\tau\Big)\,ds \\
&\leq 2\|A_0\|_\infty^2 \exp\Big(2\int_0^{2p_0+2} \sigma(\tau)d\tau\Big)\exp\Big(-2\int_0^{2n} \sigma(\tau)d\tau\Big). \qquad \blacksquare
\end{aligned}$$

The important thing that must be noted is that p_0 has a great effect on the estimate. We obtain: the bigger $\|A_0\|_\infty$ is, the bigger p_0 is.

5.2 THE STUDY OF THE DECREASE OF THE ENERGY OF WEAK SOLUTIONS. We prove (1.18). Fix $p \geq 2$ and consider

$$\forall s \in (0, T_p^{-1}), \quad A_0(s) = \left(\frac{1}{s\ln_1(s^{-1})\ln_2(s^{-1})\cdots\ln_{p-1}(s^{-1})(\ln_p(s^{-1}))^2}\right)^{1/2},$$

and $A_0(s) = 0$ in $(-1,0)\cup(T_p^{-1},1)$. We verify that $A_0 \in L^2(0,1)$. We use several times the change of variables $z = \ln s$ to get

$$\begin{aligned}
\int_{-1}^{1} A_0(s)^2\,ds &= \int_0^{T_p^{-1}} \frac{ds}{s\ln_1(s^{-1})\ln_2(s^{-1})\cdots\ln_{p-1}(s^{-1})(\ln_p(s^{-1}))^2} \\
&= \int_{\ln T_p}^{+\infty} \frac{dz}{z\ln_1(z)\ln_2(z)\cdots\ln_{p-2}(z)(\ln_{p-1}(z))^2} \\
&= \cdots = \int_{\ln_p(T_p)}^{+\infty} \frac{dz}{z^2} = \frac{1}{\ln_p(T_p)}.
\end{aligned}$$

We introduce

$$\mathcal{S}_n := \int_0^{2n} \sigma(\tau)\,d\tau.$$

Set $\alpha > 0$ and define for n large enough

$$s_n = \frac{\alpha}{\mathcal{S}_n^2 \ln_1(\mathcal{S}_n) \ln_2(\mathcal{S}_n) \cdots \ln_{p-1}(\mathcal{S}_n)(\ln_p(\mathcal{S}_n))^2}.$$

We easily see that

$$A_0(s_n)^2 \sim \frac{\mathcal{S}_n^2}{2\alpha} \quad \text{as } n \to +\infty.$$

Therefore, if we choose for example $\alpha := \frac{1}{4}$, for n large enough we have

$$\forall s \in (0, s_n), \quad A_0(s) \geq A_0(s_n) \geq S_{n-2}(s) + \frac{2 + \sigma_{n-1}(s)}{2},$$

and so

$$\forall s \in (0, s_n), \quad |A_n(s)| = A_0(s) - S_{n-1}(s).$$

Therefore

$$\begin{aligned} E(2n) &= \int_{-1}^{1} A_n(s)^2 \, ds \geq \int_0^{s_n} A_n(s)^2 \, ds = \int_0^{s_n} (A_0(s) - S_{n-1}(s))^2 \, ds \\ &\geq \int_0^{s_n} \frac{1}{2} A_0(s)^2 - S_{n-1}(s)^2 \, ds \\ &\geq \frac{1}{2} \frac{1}{\ln_p(s_n^{-1})} - (\frac{1}{2}\mathcal{S}_n + \sigma(0))^2 s_n \sim \frac{1}{2} \frac{1}{\ln_p(s_n^{-1})} \quad \text{as } n \to +\infty. \end{aligned}$$

Thus, we obtain that for n large enough,

$$E(2n) \geq \frac{1}{8} \frac{1}{\ln_p(\int_0^{2n} \sigma(\tau)\, d\tau)} \geq \frac{1}{\ln_{p-1}(\int_0^{2n} \sigma(\tau)\, d\tau)}. \qquad \blacksquare$$

At last we show that the energy of weak solutions goes to zero at infinity.

5.3 Strong stability for weak solutions. There is a way to prove that weak solutions go to zero which avoid any computation. Given $(u^0, u^1) \in V \times L^2(0,1)$ and $s_0 \in (-1,1)$, it is sufficient to prove that the sequence $(A_n(s_0))_n$ goes to zero when n goes to infinity. For proving this, we consider a new time-dependent problem; the advantage of this method is that it gives an estimate of the decrease of $(A_n(s_0))_n$. Define $\tilde{\sigma}$: $\mathbb{R}_+ \to \mathbb{R}_+$ a nonincreasing function of class C^1 such that for all $n \in \mathbb{N}$, $\tilde{\sigma}(2n+1+s) = \sigma(2n+1+s_0)$ on $(s_0 - \delta, s_0 + \delta)$ where $\delta > 0$ is small enough such that $(s_0 - \delta, s_0 + \delta) \subset (-1,1)$. Next define $\tilde{g}$: $\mathbb{R} \to \mathbb{R}$ an odd increasing function of class C^1 such that $\tilde{g}(v) = g(v)$ on $(-3|A_0(s_0)|, 3|A_0(s_0)|)$ and $\tilde{g}$ has a linear growth on $(3|A_0(s_0)|, +\infty)$. Finally, define $(\tilde{u}^0, \tilde{u}^1)$ such that the corresponding function $\tilde{A}_0$ is constant and equal to $A_0(s_0)$ on $(-1,1)$, and consider the solution $\tilde{u}$ of the following problem:

$$(5.3)\qquad \begin{cases} \tilde{u}'' - \tilde{u}_{xx} = 0 & \text{in } (0,1) \times \mathbb{R}_+, \\ \tilde{u}(0,t) = 0 & \text{on } \mathbb{R}_+, \\ \tilde{u}_x(1,t) + \tilde{\sigma}(t)\,\tilde{g}(\tilde{u}'(1,t)) = 0 & \text{on } \mathbb{R}_+, \\ \tilde{u}(0) = \tilde{u}^0, \quad \tilde{u}'(0) = \tilde{u}^1. \end{cases}$$

Since Hyp. 3 is satisfied, we deduce from Theorem 1 that the energy of $\tilde{u}$ (which we denote $E_{\tilde{u}}$) goes to zero. Then we easily deduce from (4.1) and from our definitions of $\tilde{\sigma}$ and $\tilde{g}$ that

$$2\delta A_n(s_0)^2 = \int_{s_0-\delta}^{s_0+\delta} \tilde{A}_n(s)^2\, ds \le E_{\tilde{u}}(2n).$$

Hence $A_n(s_0) \to 0$ as $n \to +\infty$. ∎

Note that we never used the specific form of the function g: this result of strong stability holds true if g is an increasing odd function of class C^1 on a neighborhood of zero, and no growth condition at infinity is required.

References

[1] M. Aassila, *On a quasilinear wave equation with a strong damping,* Funkcialaj Ekvacioj **41** (1998), 67–78.

[2] C. Bardos, G. Lebeau and J. Rauch, *Un exemple d'utilisation des notions de propagation pour le contrôle et la stabilisation des problèmes hyperboliques,* in *Proceedings of the Workshop on Non-Linear Hyperbolic Equations in Applied Sciences* (N. Bellono, ed.), Rendiconti del Seminario Matematico, Universita'e Politecnico Torino (Fascicolo Speciale 1988), 1988, pp. 11–31.

[3] P. Cannarsa, V. Komornik and P. Loreti, *Well posedness and control of semilinear wave equations with iterated logarithms,* ESAIM, Control Optimization and Calculus of Variations **4** (1999), 37–56.

[4] F. Conrad, J. Leblond and J. P. Marmorat, *Stabilization of second order evolution equations by unbounded nonlinear feedback,* Proceedings of the Fifth IFAC Symposium on Control of Distributed Parameter Systems, Perpignan, June, 1989, pp. 101–116.

[5] A. Haraux, *Comportement à l'infini pour une équation des ondes non linéaire dissipative,* Comptes Rendus de l'Académie des Sciences, Paris, série A **287** (1978), 507–509.

[6] A. Haraux, *Nonlinear evolution equations,* Lecture Notes in Mathematics **841**, Springer-Verlag, Berlin–Heidelberg–New York, 1980.

[7] A. Haraux, *Stabilization of trajectories for some weakly damped hyperbolic equations,* Journal of Differential Equations **59** (1985), 145–154.

[8] A. Haraux, *L^p estimates of solutions to some nonlinear wave equations in one space dimension,* Publications du laboratoire d'analyse numérique, University P. et M. Curie, CNRS, 1995.

[9] A. Haraux and E. Zuazua, *Decay estimates for some semilinear damped hyperbolic problems,* Archive for Rational Mechanics and Analysis **100** (1988), 191–206.

[10] M. A. Horn and I. Lasiecka, *Global stabilization of a dynamic Von Karman plate with nonlinear boundary feedback,* Applied Mathematics and Optimization **31** (1995), 57–84.

[11] V. Komornik, *Exact Controllability and Stabilization. The Multiplier Method,* Wiley, Chichester and Masson, Paris, 1994.

[12] S. Kouémou Patcheu, *On the decay of solutions of some semilinear hyperbolic problems,* Panamerican Mathematical Journal **6** (1996), 69–82.

[13] I. Lasiecka, *Uniform stabilizability of a full Von Karman system with nonlinear boundary feedback,* SIAM Journal on Control and Optimization **36** (1998), 1376–1422.

[14] I. Lasiecka, *Boundary stabilization of a 3-dimensional structural acoustic model,* Journal de Mathématiques Pures et Appliquées **78** (1999), 203–232.

[15] I. Lasiecka and D. Tataru, *Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping,* Differential and Integral Equations **6** (1993), 507–533.

[16] G. Leoni, *A note on a theorem of Pucci and Serrin,* Journal of Differential Equations **113** (1994), 535–542.

[17] J. L. Lions, *Contrôlabilité exacte et stabilisation de systèmes distribués,* Vol. 1, Masson, Paris, 1988.

[18] P. Martinez, *Decay of solutions of the wave equation with a local highly degenerate dissipation,* Asymptotic Analysis **19** (1999), 1–17.

[19] P. Martinez, *A new method to obtain decay rate estimates for dissipative systems with localized damping,* Revista Matemática de la Universidad Complutense de Madrid **12** (1999), 251–283.

[20] P. Martinez, *A new method to obtain decay rate estimates for dissipative systems,* ESAIM, Control Optimization and Calculus of Variations **4** (1999), 419–444.

[21] P. Martinez and J. Vancostenoble, *Exponential stability for the wave equation with weak nonmonotone damping,* Portugaliae Mathematica, to appear.

[22] M. Nakao, *Asymptotic stability of the bounded or almost periodic solution of the wave equation with a nonlinear dissipative term,* Journal of Mathematical Analysis and Applications **58** (1977), 336–343.

[23] M. Nakao, *Energy decay for the wave equation with a nonlinear weak dissipation*, Differential and Integral Equations **8** (1995), 681–688.

[24] M. Nakao, *Decay of solutions of the wave equation with a local nonlinear dissipation*, Mathematische Annalen **305** (1996), 403–417.

[25] M. Nakao, *On the time decay of solutions of the wave equation with a local time-dependent nonlinear dissipation*, Advances in Mathematical Sciences and Applications **7** (1997), 317–331.

[26] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, 1983.

[27] P. Pucci and J. Serrin, *Precise damping conditions for global asymptotic stability for nonlinear second order systems*, Acta Mathematica **170** (1993), 275–307.

[28] P. Pucci and J. Serrin, *Precise damping conditions for global asymptotic stability for nonlinear second order systems, II*, Journal of Differential Equations **113** (1994), 505–534.

[29] P. Pucci and J. Serrin, *Asymptotic stability for nonautonomous dissipative wave systems*, Communications on Pure and Applied Mathematics **49** (1996), 177–216.

[30] P. Pucci and J. Serrin, *Local asymptotic stability for dissipative wave systems*, Israel Journal of Mathematics **104** (1998), 29–50.

[31] C. S. Quiroga de Caldas, *On an elasticity system with coefficients depending on the time and mixed boundary conditions*, Panamerican Mathematical Journal **7** (1997), 91–109.

[32] L. R. Tcheugoué Tébou, *Stabilization of the wave equation with localized nonlinear damping*, Journal of Differential Equations **145** (1998), 502–524.

[33] J. Vancostenoble, *Optimalité d'estimations d'énergie pour une équation des ondes amortie*, Comptes Rendus de l'Académie des Sciences, Paris, Série I Mathématique **328** (1999), 777–782.

[34] J. Vancostenoble and P. Martinez, *Optimality of energy estimates for a damped wave equation with polynomial or non polynomial feedbacks*, SIAM Journal on Control and Optimization, to appear.

[35] E. Zuazua, *Stability and decay for a class of nonlinear hyperbolic problems*, Asymptotic Analysis **1** (1988), 1–28.

[36] E. Zuazua, *Uniform stabilization of the wave equation by nonlinear boundary feedback*, SIAM Journal on Control and Optimization **28** (1990), 466–478.

[37] E. Zuazua, *Exponential decay for the semi-linear wave-equation with locally distributed damping*, Communications in Partial Differential Equations **15** (1990), 205–235.